1 Introduction

1.1 The issue

This paper considers the identification of a structural linear equation using quantile regression in the presence of endogeneity problems. Since the seminal work by Koenker and Bassett (1978), there are two trends in the literature on quantile regressions when dealing with endogeneity. The first one, denoted the 'structural setting,' corresponds to models directly specified in terms of the conditional quantile of the structural equation of interest. In that case, semiparametric restrictions needed for identification are directly imposed on the structural errors terms, or on the structural model if there are no separable errors. The second trend, denoted the 'fitted-value setting,' is based on the conditional quantile of the reduced-form equation. Accordingly, the restrictions are imposed on the reduced-form errors when they can be separated. In this later setting, the analysts substitute the endogenous regressors with the fitted-values obtained from ancillary equations based on some exogenous variables. The fitted-value setting corresponds to quantile restrictions on the reduced form. As noted in Blundell and Powell (2006): "The reduced form for y_t may be of interest if the values of IVs are control variables for the policy maker." The fitted-value setting has also algebraic and computational advantages. However, the reduced form can also be viewed as an intermediary stage for calculations. Amemiya (1974) pointed out that, while substitution of fitted values in nonlinear structural functions generally yields inconsistent estimates of the structural parameters, consistent estimation methods that substitute fitted values into the structural function can rely on linearity of the regression, where the model is based on the reduced form error, with similar stochastic properties to the structural error. Even in the context of linear models for quantile regressions, it is believed that this setting corresponds to constant effect models, a little attractive characteristics. That is: it is believed that all coefficients, except the intercept, must be the same for all quantiles, as discussed in Lee (2007, p1138): " 'the fitted value' approach, which is developed by Amemiya (1982) and Powell (1983), replaces X with the fitted value of $\mu + Z'\pi$ in the system: $Y = X\beta(\tau) + Z_1'\gamma(\tau) + U$ and $X = \mu(\alpha) + Z'\pi(\alpha) + V$. To see how the fitted value approach works, consider the reduced-form equation for Y: $Y = \beta\left[\mu + Z'\pi\right] + Z_1'\gamma + \eta$, where $\eta = U + \beta V$. In order to estimate β and γ consistently, the fitted value approach requires that $Q_{\eta|Z}(\tau|z)$ be independent of z."[1] The latter corresponds to the constant effect case for the second stage of the estimated quantile regression model. This criticism has not been addressed in the recent literature, other than by falling back to the structural setting, or by assuming constant effect for the true quantile process. We deal with this gap. In this paper, we exhibit a particular case of non-constant effect (i.e., quantile-dependent coefficients) for linear quantile regression with the fitted-value setting.

The literature on the structural setting for linear quantile regressions is abundant,[2] while it meets computation costs for correcting endogeneity issues.[3] In contrast, the fitted-value setting corresponds to a simple two-step quantile regression procedure, analogous to the 2SLS method, and has been readily employed by empirical researchers who are not always expert econometricians or programmers, conveniently

[1] We do not make explicit the notations here, as they are rather obvious and this is only a citation. However, the reader in doubt may refer to Lee's paper for full details.

[2] See for example: Abadie et al. (2002), Chen et al. (2003), Chesher (2003), Hong and Tamer (2003), Chernozhukov and Hansen (2005, 2006, 2008), Imbens and Newey (2006), Ma and Koenker (2006), Chernozhukov, Imbens and Newey (2007), Lee (2007), Jun (2009), Chernozhukov, Fernandez-Val and Kowalski (2015).

[3] As documented in Kim and Muller (2013).

avoiding computation burden.[4] Namely, with the fitted-value setting, no control function nonparametric estimation, no simulations, no computation iteration or grid are necessary. Partial theoretical results had been obtained by Amemiya (1982) and Powell (1983), who analyse the two-stage least-absolute-deviations estimators in simple settings, and by redefining the dependent variable. Chen and Portnoy (1996) and Kim and Muller (2004, 2015) investigate such two-stage quantile regressions with diverse first-step estimators (LS, LAD, and trimmed least squares estimators) and in general settings.

However, according to Lee (2007)'s analysis, all these authors deal with constant effect specifications. In contrast, our focus in this paper is the occurrence of non-constant effect with the fitted-value setting, although heterogeneity will still not be allowed for a subset of model coefficients (for endogeneous or exogenous variables, depending on hypotheses).

For this, we first show that any separable model can be made to satisfy a quantile restriction for any quantile θ, provided it allows for an inconsistency term, which we characterize. Second, we show how the influence of the inconsistency term can be weakened in terms of its link with covariate effects. This is done by assuming some weakened IV conditions, which may even allow for endogenous regressors in the reduced form. Under these new IV conditions, we show that non-constant effect can arise in linear quantile regression even under endogeneity dealt with the fitted-value setting. However, in that case, only the constant effect coefficients can be identified. Finally, we show that the constant effect (respectively, non-constant effect) coefficients of the reduced form can be simply transmitted to constant effect (respectively, non-constant effect) coefficients in the structural equation.

[4] Arias et al. (2001), Garcia et al. (2001), Chortareas et al. (2012) and Chepatrakul et al. (2012).

1.2 Practical examples

We now illustrate our reflections with a few economic examples in which constant effect occurs at least for some variables in the regressions. A first illustration is the assessment of a policy rule that, first, is constant for an identifiable population, and second, affects an outcome variable that is shifted by a quantity proportional to the policy rule. For example, one may wish to assess the impact of a cash transfer program on family earnings. Let y_t be the total earnings of family t and Y_t be the policy treatment, which is here a dummy variable describing whether or not the family is covered by the program. Assume that the treated families can be identified through some observable characteristics, for example, because they live in a specific place or have given socio-demographic characteristics. For example, in France, family allowances are implemented through cash transfers that are exclusively based on the number of children by age class in the family (ADECRI, 2008). The information on family composition is generally observed in household surveys, which could be used to run a regression of family earnings. In that case, the transfer amount, which is the constant parameter γ to estimate, is generally not or is ill observed in the surveys. However, the treatment dummy variable (Y_t) can be observed, while it may be endogenous. For example, families with unobserved low motivation for work may have more children and lower income. This may be the case if having children is used by some families to access social aid and compensate for insufficient incomes.

Another illustration is the assessment of the impact on household income of taxes calculated from observable categories of household. Assume that these unobserved tax amounts are fixed within each household category. Then, when running a quantile regression with the household net income as a dependent variable, the taxes can be described by a vector of fixed coefficients γ that measures the role of the dummy variables specifying the household categories. Another example is that of racketing extraction tariffs imposed by mafia organisations, which have the same property of being fixed for some given categories of businesses, while their actual amounts are generally unobservable by researchers.

A final illustration is the assessment of the contribution to total family expenditure of some unobserved expenses for a discrete good, or a discrete service, when its price is fixed (e.g, when provided by public institutions). For example, the acquisition or the ownership of a given durable good with fixed characteristics might be observable, but not its fixed price. Estimating the corresponding coefficient γ would allow some inference about the unobserved price.

In all of these illustrations, we have constant effect for a treatment variable of interest. In these models, other variables x_{1t} that determine the studied outcomes may be included.[5] In the considered examples, their coefficients β should generally correspond to non-constant effect. Indeed, the economic theory does not provide any reason for imposing constant effect for these variables. Such non-constancy is likely to generate a specification bias in quantile regressions that would wrongly assume constant coefficients for these variables. However, if the interest of the researcher is exclusively in the coefficients γ of the structural endogenous regressors, the fact that the coefficients β of the structural exogenous regressors have non-constant effect that cannot be identified in the fitted-value setting is not an issue as long as constant effect coefficients can be identified, which we show in this paper.

Finally, some observed variability in the treatment can easily be incorporated in these cases by interacting the treatment variable with some observed characteristics, or by considering subpopulations defined in terms of these characteristics. This is an easy way to relax too stringent specifications of constant effect.

[5] We shall be able to relax the exogeneity assumption later on.

The paper is organized as follows. Section 2 presents the model and the assumptions. In Section 3, we exhibit and analyse a case of non-constant effect for the fitted-value setting. Finally, we conclude the paper in Section 4.

2 The Model

Assume that our interest lies in the parameter vector $\alpha_\theta \equiv (\beta'_\theta, \gamma'_\theta)'$ in the following linear equation for T observations and an arbitrary quantile index $\theta \in (0, 1)$ that will denote quantile restrictions introduced later on.

$$y_t = x'_{1t}\beta_\theta + Y'_t\gamma_\theta + u_{t\theta} = z'_t\alpha_\theta + u_{t\theta}, \tag{1}$$

where $[y_t, Y'_t]$ is a $(G+1)$-row vector of endogenous variables, x'_{1t} is a K_1-row vector of exogenous variables, $z_t = [x'_{1t}, Y'_t]'$ and $u_{t\theta}$ is an error term.

Since we wish to study non-constant effect models, we emphasize that the coefficient vector and the errors may vary with the considered quantile index θ. We denote by x'_{2t} the row vector of the K_2 exogenous variables excluded from (1), and we assume $K_2 \geq G$. We further assume that Y_t can be linearly predicted from the exogenous variables through the following equation, which we assume to be correctly specified.

$$Y'_t = x'_t\Pi + V'_t, \tag{2}$$

where $x'_t = [x'_{1t}, x'_{2t}]$ is an unbounded K-rows vector with $K = K_1 + K_2$, Π is a $K \times G$ matrix of unknown parameters, and V'_t is a G-row vector of unknown error terms. Again, some stochastic assumptions on the errors V_t must be made so as to complete (2) for defining a correctly specified model. For example, to fix ideas, one may assume that the conditional expectation of V_t is zero, as for OLS estimation, thus ensuring the consistency of the fitted-value for Y_t; or alternatively some quantile restriction at a quantile θ as in Kim and Muller (2004). To avoid absurdities, we assume that the columns in x_t are linearly independent. Using (1) and (2), y_t can also be expressed as:

$$y_t = x_t'\pi_\theta + v_{t\theta}, \tag{3}$$

where

$$\begin{aligned} \pi_\theta &= H(\Pi)\alpha_\theta \text{ with } H(\Pi) = \left[\begin{pmatrix} I_{K_1} \\ 0 \end{pmatrix}, \Pi \right] \\ \text{and } v_{t\theta} &= u_{t\theta} + V_t'\gamma_\theta. \end{aligned} \tag{4}$$

Again here, we allow for the vector of coefficients π_θ to vary with the quantile θ. We first consider the following quantile restriction on the reduced-form errors for a given quantile θ_0.

Assumption 1: *$E(\psi_{\theta_0}(v_{t\theta_0})|x_t) = 0$, for an arbitrary given $\theta_0 \in (0,1)$, where $\psi_\theta(z) \equiv \theta - 1_{[z\leq 0]}$, and $1_{[.]}$ is the indicator function.*

Assumption 1 imposes that zero is the given θ_0^{th}-quantile of the conditional distribution of $v_{t\theta_0}$. This assumption allows the identification of the coefficients α_θ of the quantile regression model centered in quantile θ_0. It is associated with the fitted-value setting in which, first, the conditional quantile restriction is placed on the reduced-form error $v_{t\theta_0}$, and second, the information set used for the conditional restriction exclusively consists of the exogenous variables x_t. It has been used in Amemiya (1982), Powell (1983), Chen and Portnoy (1996), and Kim and Muller (2004, 2015). In particular, Kim and Muller (2015) provide an analysis of asymptotic properties based on this assumption and with broad stochastic conditions. One issue is how quantile restrictions for different quantiles come together so as to define a unique quantile process for a single model; that is, so that all these restrictions are compatible. The analysis we pursue will clarify this point, which is at the core of understanding the possible non-constancy of effects.

The link of structural and reduced-form parameters is described by (4). Identification of the structural parameters is obtained from the following assumption.

Assumption 2: $H(\Pi)$ *is of full column rank.*

Assumption 2 is the usual necessary condition for IV regressions, for example, obtained through usual exclusion restrictions. This is an identification condition for simultaneous linear equations models. A first-stage estimator of Π in (2), $\hat{\Pi}$, allows the construction of the fitted value $\hat{Y}_t' = x_t'\hat{\Pi}$, which is substituted for Y_t' in the final estimation stage. We define, for any quantile θ, the two-stage quantile regression estimator $\hat{\alpha}_\theta$ of α_θ as a solution to:

$$\min_{\alpha} \; S_T(\alpha, \hat{\Pi}, \theta) = \sum_{t=1}^{T} \rho_\theta(y_t - x_t' H(\hat{\Pi})\alpha), \text{ where } \rho_\theta(z) = z\psi_\theta(z) \tag{5}$$

In the next section, we exhibit some non-constant effect with the fitted-value setting.

3 Non-Constant Effect in the Fitted-Value Setting

3.1 Regularities and quantile restrictions

Let us start again with Equations (1) and (2), with possible non-constant effect in structural and reduced-form equations, *but without a priori imposing Assumption 1.* In order to deal with unique quantile values so as to simplify the discussion, we make the following continuity and monotonicity assumption, for a starting value θ_0 of the quantile index.

Assumption 3: *For a given quantile index θ_0, the cdf of $v_{t_{\theta_0}}$ conditional on x_t, denoted $F_{v_{t\theta_0}|x_t}$, the cdf of $v_{t\theta_0}$ conditional on x_{1t}, denoted $F_{v_{t\theta_0}|x_{1t}}$, and the marginal cdf of x_{2t}, denoted $F_{x_{2t}}$, are continuous and strictly increasing.*

First, note that, under Assumption 3, an inverse cdf term can always be isolated in the reduced-form equation, for θ_0, by denoting: $y_t = x_t'\pi_{\theta_0} + v_{t\theta_0} = F^{-1}_{v_{t\theta_0}|x_t}(\theta) + x_t'\pi_{1\theta_0} + v^*_{t\theta}$ and $v^*_{t\theta} \equiv v_{t\theta_0} - F^{-1}_{v_{t\theta_0}|x_t}(\theta)$. Then, by construction, $v^*_{t\theta}$ is characterized by the conditional quantile restriction: $E(\psi_\theta(v^*_{t\theta})|x_t) = \theta - P\left[v^*_{t\theta} \leq 0|x_t\right] = \theta - P\left[v_{t\theta_0} \leq F^{-1}_{v_{t\theta_0}|x_t}(\theta)|x_t\right] = \theta - \theta = 0$. As a consequence, one can always obtain a quantile regression restriction at θ, even distinct from θ_0, for the reduced-form, provided one accepts a possible nuisance inconsistency term $F^{-1}_{v_{t\theta_0}|x_t}(\theta)$ that can affect all the coefficients of the model. Note that $v^*_{t\theta}$ depends both on θ and on θ_0. In the next subsection, we weaken the quantile restriction at quantile index θ_0, so as to allow enough flexibility for generating non-constant effect.

3.2 Generating non-constant effect

Let us now return to our maintained model, but *instead of Assumption 1*, we now consider the following weaker restriction.

Assumption 4: *For a given quantile θ_0 and under Assumption 3:*

$$F^{-1}_{v_{t\theta_0}|x_t}(\theta) = F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta), \tag{6}$$

for all $\theta \in (0, 1)$.

The next proposition, of which proof is in the appendix, translates this assumption in terms of conditional independence under Assumption 3.

Proposition 1: *Under Assumption 3, Assumption 4 is equivalent to:* $v_{t\theta_0}$ *is independent on* x_{2t}*, conditionally on* x_{1t}*.*

Assumption 4 is akin to conditions in the control function literature, with here the control function known to depend on x_{1t} only. We do not require complete exogeneity, as we will discuss later. Assumption 4 is also related to the notion of 'conditional exogeneity' in White and Chalak (2010). Note that the restriction in Assumption 4 implies that the OLS in the reduced form may be inconsistent in the allowed case where the x_{1t} are endogenous. One may also have $E(\psi_\theta(v_{t\theta})|x_{1t}) \neq 0$ under this hypothesis. This means that x_{1t} may be endogenous in the sense of the quantile regressions of quantile θ for equation (3). In such situation, equation (3) no longer characterizes a typical 'reduced form' based only on exogenous regressors, although to simplify we still denote it the '*reduced-form equation.*' In the remainder of this section, we show how non-constant effect can be obtained for conditional quantiles of the reduced-form, and then conveyed to the conditional quantiles of the structural form.

An example is a structural wage equation for a labour market study for a sample of workers, in which the dependent variable (y_t) is the logarithm of individual wage rate, while the two independent variables in this equation are the industrial sector dummy (x_{1t}) and the workers's education level (Y_t), and the regressors in the reduced form are again the industrial sector dummy and the worker's birth quarter (x_{2t}). The birth quarter is used as an instrument for her education level that is typically assumed to be the sole endogenous independent variable in the structural model (e.g., in Angrist and Krueger, 1991, or in Angrist and Pischke, 2009, who find constant effect for this variable). However, in such an empirical problem, one may expect the log wage rate to be positively correlated with the capital of the firm, omitted from the model, while incorporated in the error v_t, and which should be typically correlated with the industrial sector dummy. Then, x_{1t} and v_t may be correlated, while x_{2t} and v_t should be independent according to the usual justification for using quarter of birth as an instrument in wage equations.[6]

Since $v_{t\theta} = u_{t\theta} + V'_{t\theta}\gamma_\theta$, Assumption 4 for all θ can also be obtained, by assuming that $u_{t\theta}$ and V_t are independent of x_{2t}, conditional on x_{1t}. Thus, this condition can also be seen as the consequence of a natural, while strong, instrumental variable characterization of x_{2t} for the structural model.

We now show that Assumption 4 implies that there is constant effect in the quantile regressions of the reduced-form equation for the coefficients of the variables in x_{2t}, but not necessarily for the coefficients of the variables in x_{1t}.

Proposition 2: *Under Assumption 3 and 4, for a quantile regression process of the reduced form* (3)*:*

(a) There is constant effect for the variables in x_{2t}.

(b) Non-constant effect is possible for the variables in x_{1t}.

(c) For all θ, $F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta)$ *must be linear in* x_{1t}.

The proof is in the Appendix. Result *(c)* is implied in particular by the often used 'linear location-scale hypothesis' in the quantile regression literature on non-constant effect (e.g., Card and Lemieux, 1996, and Koenker, 2005, pp 104-105). In cases where Result *(c)* would be judged a too unrealistic consequence, this can be easily relaxed by incorporating polynomial terms in x_{1t}, or splines, in the model, as is usual for approximating nonlinear functions. Alternatively, one could specify a reduced form

[6] Conditioning on the industrial sector might make the hypothesis of independence of the birth quarter and the error more plausible if the sector was a common determinant of the latter two variables, although it is unclear why this should be the case.

(3) as being partially linear in x_{2t}, and possibly nonlinear in x_{1t}, with an unknown nonlinear functional form. In that case, the above reasoning delivering constant effects for x_{2t} and unrestricted (nonlinear) effect for x_{1t} would remain valid, and Result *(c)* would not be necessary. However, this would push us toward nonparametric estimation methods, which is not what we discuss in this paper. Finally, instead of imposing Assumption 4, one may first test for which coefficients the hypothesis of constant effect is not rejected in a typical quantile regression estimation. This would guide the specification of Assumption 4 in the considered data. In the next subsection, we discuss the identification of the reduced-form parameters under Assumption 4.

3.3 Identification

The next proposition characterizes the identification of quantile regression estimators under Assumptions 3 and 4.

Proposition 3: *Under Assumptions 3 and 4, which imply the linearity property ($F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta) = x'_{1t}\pi_{1F\theta}$), the non-identification in the θ^{th} conditional quantile regression estimator of the reduced-form linear model (3) can be confined to the first K_1 variables, for any quantile index θ possibly distinct from θ_0. That is:, the coefficients $\pi_{2\theta} = \pi_2$ of the x_{2t} variables in the reduced form are identified.*

The proof is in the Appendix. The reason for the results in Proposition 3 is that under v_t independent on x_{2t} conditionally on x_{1t}, only the coefficient of x_{2t} can be identified through solving these K_2 orthogonality conditions, even conditionally on x_{1t}.

In these conditions, as seen before, non-constant effect is possible for the conditional quantiles of the reduced form for $\pi_{1\theta}$ for any θ, even though Assumption 4 is made only for the value θ_0. There is no constant effect for π_2 that has been shown not to vary with θ. For example, a special case of non-constant effect quantile regression can be parameterized such as: $\pi_{1\theta} = \pi_1 + \theta$. However, even though the exhibited non-constant effect in the reduced form allows for generalization of the model as compared to constant effect cases, the non-constant effect parameters cannot be identified with usual quantile regression settings. Though, the other parameters of the model can be consistently estimated.

One may also want to consider fully specified quantile regressions, while only at some quantiles. In that case, one may require Assumption 1 for these quantiles only, keeping Assumption 4 for other quantiles. Jun (2008) studies variations across quantiles of identification through instrument variables. This suggests an interest in applying different semi-parametric IV restrictions at different quantiles. This would allow for many distinct models with (partial) non-constant effect.

3.4 Transmitting the non-constant effect to the structural form

We now assess the consequences for the structural model of the partial occurrence of non-constant effects in the reduced form. As we discussed before, Assumption 4 implies that the parameter $\pi_\theta = (\pi'_{1\theta}, \pi'_{2\theta})'$ in the reduced form model (3), where $v_{t\theta}$ satisfies $E(\psi_\theta(v_{t\theta})|x_t) = 0$, is such that $\pi_{2\theta}$ is identified and corresponds to constant effect, while $\pi_{2\theta}$ is not identified and may allow for non-constant effect.

Proposition 4: *Under Assumptions 3 and 4, for the stuctural quantile model (1) with the fitted-value setting based on the predictive equation (2):*

(a) The coefficient vector γ_θ of the endogenous regressors in the structural model does not vary with the quantile index θ: $\gamma_\theta = \gamma$ for all $\theta \in (0,1)$. Endogenous

variables have constant effects and the coefficient vector γ is identified.

(b) The coefficient vector β_θ of the exogenous regressors in the structural model can vary with the quantile index θ. Exogenous variables may have non-constant effect. However, the coordinates of β_θ exhibit exactly the same unknown inconsistency term as their respective coefficients $\pi_{1\theta}$ in the reduced-form model. The vector β_θ is not identified in general.

The proof is in the Appendix. When a first-stage estimation is performed based on (2), the independent variables x_t consists of vectors x_{1t} and x_{2t}. If a non-zero asymptotic inconsistency term is present only in the coefficients of x_{1t} in the reduced-form estimator $\hat{\pi}_\theta$, which is the case on which we focus, then the non-zero asymptotic inconsistency in the second-stage estimator $\hat{\alpha}_\theta = [\hat{\beta}'_\theta, \hat{\gamma}'_\theta]'$ is exclusively confined to the coefficients of x_{1t}; that is, only β_θ is not identified. Therefore, the parameter (γ_θ) for the endogenous variable Y_t in the structural equation in (1) can be identified. In this setting, because π_2 is characterized by constant effect, and because γ_θ is not connected to $\pi_{1\theta}$, we have also constant effect for γ_θ. In contrast, a non-constant effect may occur for β_θ.

Allowing for the weakened IV condition in Assumption 4 has enabled us to introduce non-constant effect on the vector β_θ, even though this parameter is not identified. This alone is a generalization of the *stricto sensu* constant effect structural quantile regression, which may be useful if the researcher's interest is concentrated on vector γ that is identified and can be estimated consistently. Indeed, this setting avoids misspecification of the quantile regression, if the true DGP involves non-constant effect for β_θ and constant effect for γ.

4 Conclusion

In this paper, we have shown how some particular cases of non-constant effect can be obtained with two-stage quantile regressions based on the fitted-value setting under endogeneity. However, we have also established that only the coefficients of constant-effect variables can be identified, even though non-constant effect is present for the other variables. Finally, we have discussed a few practical examples where our approach would be useful.

Our results are based on relatively little demanding instrumental variable conditions, for example that the reduced-form errors (or the structural errors) are independent of *SOME excluded variables,* conditionally on the other independent variables. Such weakening of the usual IV conditions is potentially useful since convincing instruments are typically difficult to find. Then, any reduction in the set of necessary instruments is valuable.

Our approach is also interesting because a quantile regression model with constant effect for one coefficient only (or a few coefficients only) is much more general that a general constant effect for all coefficients of a quantile regression model. Thus, our approach corresponds to a specification that is intermediate between the constant effect quantile model and the fully non-constant effect quantile model.

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Proofs:

Proof of Proposition 1: Let us first consider Assumption 4 without the variables x_{1t}, that is, $F^{-1}_{v_{t\theta_0}|x_{2t}}(\theta) = F^{-1}_{v_t\theta_0}(\theta)$, for all θ, which we wish to show to be equivalent to: $v_{t\theta_0}$ is independent of x_{2t}. By definition, under Assumption 3, the latter statement means that $f_{v_{t\theta_0},x_{2t}}(v,w) = f_{v_{t\theta_0}}(v)\ f_{x_{2t}}(w)$, for all v and w, and where $f_{v_{t\theta_0},x_{2t}}$ is the joint pdf of $v_{t\theta_0}$ and x_{2t}, $f_{v_{t\theta_0}}$ is the marginal pdf of $v_{t\theta_0}$, and $f_{x_{2t}}$ is the marginal pdf of x_{2t}. Since, under Assumption 3, there is no level of x_{2t} with vanishing marginal pdf, we can rewrite the equality as $f_{v_{t\theta_0}}(v) = f_{v_{t\theta_0},x_{2t}}(v,w)/f_{x_{2t}}(w)$, for all v and

w. Then, by integration with respect to v on both sides of the equality, we obtain $\theta = F_{v_{t\theta_0}}(v) = F_{v_{t\theta_0}|x_{2t}}(v \mid x_{2t} = w)$, for all w. By inversion, thanks to Assumption 3, this is equivalent to $F^{-1}_{v_{t\theta_0}|x_{2t}}(\theta) = F^{-1}_{v_{t\theta_0}}(\theta) = v$, for all v, i.e., for all $\theta = F_{v_{t\theta_0}}(v)$. Introducing variables x_{1t} and conditioning on them is straightforward by considering the different cdfs for each level of x_{1t}. QED.

Proof of Proposition 2: Let us consider *another* quantile index θ different from the previously chosen θ_0 and let us impose Assumption 1 for this θ, that is: the typical quantile restriction for a quantile θ. In that way, we can investigate the features of a quantile regression model centered on a quantile index θ. Then, we examine how this conditional quantile restriction can coexist with Assumption 4 assumed with the chosen θ_0.

The conditional quantile restriction at θ is: $P\left[v_{t\theta} \leq 0 \mid x_t\right] = \theta$, i.e., $P\left[y_t \leq x_t'\pi_\theta \mid x_t\right] = \theta$, which implies that $P\left[x_{1t}'\pi_{1\theta_0} + x_{2t}'\pi_{2\theta_0} + v_{t\theta_0} \leq x_{1t}'\pi_{1\theta} + x_{2t}'\pi_{2\theta} \mid x_t\right] = \theta$,

where $\pi_{1\theta_0}, \pi_{2\theta_0}, \pi_{1\theta}, \pi_{2\theta}$ are the respective components of π_{θ_0} and π_θ according to the partition of x_t into x_{1t} and x_{2t}. By regrouping, we obtain

$$P\left[v_{t\theta_0} \leq x_{1t}'\left(\pi_{1\theta} - \pi_{1_1\theta_0}\right) + x_{2t}'\left(\pi_{2\theta} - \pi_{2\theta_0}\right) \mid x_t\right] = \theta,$$

which in turn implies that $F_{v_t|x_t}\left[x_{1t}'\left(\pi_{1\theta} - \pi_{1\theta_0}\right) + x_{2t}'\left(\pi_{2\theta} - \pi_{2\theta_0}\right)\right] = \theta$. Finally, under Assumption 3, we have

$$x_{1t}'\left(\pi_{1\theta} - \pi_{1\theta_0}\right) + x_{2t}'\left(\pi_{2\theta} - \pi_{2\theta_0}\right) = F^{-1}_{v_{t\theta_0}|x_t}(\theta) = F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta), \qquad (7)$$

where the latter equality is obtained using Assumption 4. *(a)* Since $F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta)$ does not depend on x_{2t}, which is not zero, equation (7) implies $\pi_{2\theta} = \pi_{2\theta_0}$. As a consequence, we drop the dependence of $\pi_{2\theta}$ on θ, which becomes π_2. *(b)* Since there is no restriction on the effect of the variables in x_{1t}, their coefficients may vary with θ in that case. To be consistent with (7), we must also have *(c)*, which is therefore in fact a consequence of the hypotheses. QED.

Proof of Proposition 3:

We only need to check what happens when applying Assumption 4 to the identification of the quantile regression estimator, instead of applying the complete quantile regression restrictions nin Assumption 1. Indeed, under Assumption 4, we have the following theoretical restrictions:

$$E(x_{2t}\psi_\theta(y_t - x_{2t}'\pi_{2\theta_0} - x_{1t}'(\pi_{1\theta_0} + \pi_{1F\theta}))) = 0, \tag{8}$$

while

$$E(x_{1t}\psi_\theta(y_t - x_{2t}'\pi_{2\theta_0} - x_{1t}'(\pi_{1\theta_0} + \pi_{1F\theta}))), \tag{9}$$

which is typically assumed to be zero for quantile regressions under Assumption 1, is here undetermined.

As in usual settings of quantile regressions, the restrictions of orthogonality with respect to x_{2t}, (8), are satisfied by the quantile regression estimator under Assumption 4. In contrast, the restrictions of orthogonality with respect to x_{1t}, (9), may or may not be satisfied. Therefore, there is no identification of the usual quantile regression estimator under Assumption 4. A formal way to see it is that under Assumptions 3 and 4, we have: $0 = E(\psi_\theta(v_{t\theta}^*)|x_t) = E(\psi_\theta(v_{t\theta_0} - F^{-1}_{v_{t\theta_0}|x_t}(\theta))|x_t)$

$= E(\psi_\theta(y_t - x_t'\pi_{\theta_0} - F^{-1}_{v_{t\theta_0}|x_{1t}}(\theta))|x_t) = E(\psi_\theta(y_t - x_t'\pi_{\theta_0} - x_{1t}'\pi_{1F\theta})|x_t) = 0$. This implies:

$$E(x_t\psi_\theta(y_t - x_{2t}'\pi_{2\theta_0} - x_{1t}'(\pi_{1\theta_0} + \pi_{1F\theta}))) = 0. \tag{10}$$

This is the linearity property that allows the definition of a non-constant effect $(\pi_{1\theta_0} + \pi_{1F\theta})$. In contrast, the second restriction $(E(x_t\psi_\theta(y_t - x_t'\pi_\theta)) = 0)$ defines what is estimated

Eq. (10) shows that $\pi_{2\theta_0}$ and $\pi_{1\theta_0} + \pi_{1F\theta}$ are identified for quantile regression estimation at quantile θ. Therefore, $\pi_{1\theta}$ may be unidentified. The issue is that, at that stage, one does not want to identify $\pi_{1\theta_0}$ but rather $\pi_{1\theta}$.

The reduced form for quantile index θ is: $y_t = x'_{1t}\pi_{1\theta} + x'_{2t}\pi_{2\theta} + v_{t\theta}$.

There are no reasons why $\pi_{1\theta}$ and $\pi_{1\theta_0} + \pi_{1F\theta}$ should coincide. $\pi_{1\theta_0} + \pi_{1F\theta}$ is not necessarily the parameter one want to estimate. Therefore, there should be an estimation bias in general for the coefficients of x_{1t}. Similarly, there is no reason why we should equal $v_{t\theta}$ and $v^*_{t\theta}$. QED.

Proof of Proposition 4:

We decompose the link of the reduced-form and the structural-form parameters, by splitting system (4) into two blocks of equations, partitioning $\Pi = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = [\Pi'_1, \Pi'_2]'$ according to the partition of π_θ. We obtain:

$$\pi_{1\theta} = \beta_\theta + \Pi_1\gamma_\theta \tag{11}$$

$$\pi_{2\theta} = \Pi_2\gamma_\theta, \tag{12}$$

where $\pi_{2\theta} = \pi_2$, which does not include an inconsistency term, is identified and does not depend on θ, as seen above. If the system is exactly-identified (i.e., $K_2 = G$), then γ_θ can be directly expressed in terms of Π_2 and π_2, which implies that γ_θ is identified and does not depend on θ.

For the other over-identifying case (i.e., $K_2 > G$), there are more equations than is necessary to identify γ_θ. However, similarly to 2SLS or GMM, γ_θ is still identified, providing the hypotheses made are valid, and the considered models are well specified. In that case, only G arbitrary equations in (12) can be kept to define γ_θ. When estimation is considered, one can enhance the asymptotic performance of the resulting estimators by choosing appropriate weighting matrices to combine the equations in

(12). In either the exactly-identifying case or the over-identifying case, γ_θ is a function only of Π_2 and π_2. Hence, the properties of identification and of constant effect in $\pi_{2\theta}$, without inconsistency term, is conveyed to $\gamma_\theta \equiv \gamma$.

On the other hand, since $\gamma_\theta = \gamma$ is fully determined by (12), and Π_1 is given (11) shows that $\beta_\theta = \pi_{1\theta} - \Pi_1 \gamma$ incorporates the non-constant effect from $\pi_{1\theta}$, with exactly the same non-identification problem and inconsistency term as in $\pi_{1\theta}$. Indeed, since $\pi_{1\theta}$ includes an unknown inconsistency term, $\pi_{1\theta}$ is not identified even though Π_1 and γ are identified. In that case, β_θ includes also this term and is not identified either. QED.